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Multi-instantons and Maldacena's conjecture

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Abstract: We examine certain n-point functions G_n in $\mathcal{N}=4$ supersymmetric SU(N) gauge theory at the conformal point. In the large-N limit, we are able to sum all leading-order multi-instanton contributions exactly. We find compelling evidence for Maldacena's conjecture: (1) The large-N k-instanton collective coordinate space has the geometry of $AdS_5 \times S^5$. (2) In exact agreement with type IIB superstring calculations, at the k-instanton level, $G_n = \sqrt{N}\, g^8\, k^{n-7/2} e^{-8\pi^2 k/g^2} \sum_{d|k} d^{-2} \cdot F_n(x_1, \ldots, x_n)$, where F_n is identical to a convolution of n bulk-to-boundary SUGRA propagators.

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A remarkable conjecture by Maldacena [1] (see also [2] and references therein) relates the large-N limit of 4-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) at the conformal point of vanishing VEVs, to the low-energy behavior of type IIB superstrings on $AdS_5\times S^5$. In this conjectured correspondence, the SYM theory is defined on the boundary of the AdS_5; the generating functionals of SYM correlation functions are then equated to partition functions in the bulk with specific boundary conditions. When the string coupling is small and the radii of both the AdS_5 and S^5 are large (namely small g^2 but large g^2N, where g is the gauge theory coupling) the string calculation can be approximated by classical supergravity, whereas when g^2N is small the SYM model is amenable to standard perturbative and semiclassical analysis.

An important consequence of Maldacena's conjecture is that Yang-Mills instantons should be related in a specific way to the D-instantons of the IIB string theory on $AdS_5\times S^5$. In particular, Banks and Green [3] showed that the $AdS_5\times S^5$ background solves the field equations of the same $\mathrm{SL}(2,\mathbb{Z})$ invariant effective action which describes D-instanton corrections to the IIB theory in flat space. The proposed AdS/CFT correspondence then yields predictions for an infinite series of multi-instanton contributions in the $\mathcal{N}=4$ theory. This relation was studied, at the 1-instanton level, in ref. [4] for gauge group SU(2), and in ref. [5] for general SU(N). In this Letter, we extend the 1-instanton results of [5] to arbitrary Yang-Mills multi-instantons. For any finite N, the jump from single- to multi-instantons is always accompanied by an enormous increase in calculational complexity, endemic to the ADHM formalism. But in the large-N limit there are compensating simplifications, namely:

(i) The integration over the k-instanton collective coordinate space ("ADHM moduli space") becomes dominated by k single instantons living in k mutually commuting SU(2) subgroups of SU(N). (In this one respect only, the problem becomes dilute-gas-like.) This intuitively obvious remark, which we derive below from a large-N saddle-point treatment of the ADHM formalism, follows from statistical considerations alone, and is independent of supersymmetry. Additionally in the large-N limit, other less intuitive but equally remarkable features appear, which depend crucially on the $\mathcal{N}=4$ supersymmetry:

(ii) To leading order in the large-N saddle-point expansion, the geometry of the k-instanton moduli space is described by $(AdS_5\times S^5)^k$. Here the AdS_5 factors are coordinatized by the positions and sizes (x_n^i,ρ_i) of the k instantons, as has been noted previously [4, 6]. But the emergence of the S^5 factors is a newly observed feature of large N even at the 1-instanton level; the $(S^5)^k$ coordinates are the auxiliary antisymmetric-tensor variables χ^i_{AB}, $1\le i\le k$, $1\le A,B\le 4$, used to bilinearize a certain fermion quadrilinear term in the $\mathcal{N}=4$ instanton

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action. One can check that the $SO(6)_R$ non-singlet operators which correspond to the Kaluza-Klein modes of the IIB dilaton on S^5 couple to these angular degrees of freedom in the correct way [7].

(iii) Integrations in the vicinity of the large-N saddle-point generate an attractive singular potential between the k single instantons that draws them all to a common point, thereby reducing the moduli space from k copies to a single copy of $AdS_5 \times S^5$. (In this respect the problem is very *un*-dilute-gas-like.)

(iv) At the k-instanton level, the leading-order small-fluctuations effective lagrangean expanded about this reduced moduli space turns out to be identical to 10-dimensional $\mathcal{N}=1$ supersymmetric SU(k) Yang-Mills theory on flat space, dimensionally reduced to $0+0$ dimensions. The collective coordinate measure thus factorizes into the measure on $AdS_5 \times S^5$, times the partition function $\mathcal{Z}_k$ for this SU(k) theory. It is remarkable the latter factor is precisely the partition function that describes D-instantons in *flat* space [8].

In what follows we will focus, for definiteness, on certain gauge-invariant n-point functions $G_n(x_1, \ldots, x_n)$ ($n = 16, 8$ or 4) of chiral superfield components that were defined in ref. [4]. At the conformal point the SYM model has 16 exact, unlifted, adjoint fermion zero modes (two supersymmetric plus two superconformal modes, times four supersymmetries) that must be saturated by explicit field insertions in the correlator. In G_{16} each such fermion is inserted once at a distinct space-time point, whereas in G_8 and G_4 the relevant insertions consist of 8 fermion-bilinear and 4 fermion-quadrilinear pieces of bosonic operators, respectively. The G_n provide a way to compare the Yang-Mills and SUGRA sides of Maldacena's conjecture. On the SYM side, G_n is given semiclassically by a k-instanton collective coordinate integral; an independent calculation is required for each k. In contrast, in the D-instanton/SUGRA picture, G_n has the effective-vertex structure of n bulk-to-boundary propagators tied to a single point in $AdS_5 \times S^5$ that is integrated over; the k-instanton expansion simply consists of Taylor expanding the overall coefficient, which is proportional to the modular form $f_n(\tau, \bar\tau)$ [8]. As is emphasized in ref. [4], at the 1-instanton level $G_n(x_1, \ldots, x_n)$ has the identical propagator-like form in both pictures. In fact, we show below that this functional resemblance extends to all k, due to properties (i) and (iii). As a sharper test of Maldacena's conjecture, we calculate the leading semiclassical contribution to the coefficient of G_n from each k-instanton sector in the SYM model. Thanks to properties (i)–(iv), the collective coordinate integrals can be carried out explicitly in the limit $N \to \infty$. We find:

$$G_n(x_1, \ldots, x_n)\Big|_{k\text{-inst}} = \sqrt{N}\, g^8\, k^{n-7/2} e^{-8\pi^2 k/g^2} \sum_{d|k} d^{-2} \cdot F_n(x_1, \ldots, x_n)\,. \qquad (1)$$

Here the k-independent function $F_n(x_1,\ldots,x_n)$ looks like a convolution of n bulk-to-boundary SUGRA propagators, as already noted, while the summation term over the positive integer divisors of k comes from a recent evaluation of $\mathcal{Z}_k$ in refs. [8, 9, 10].

The SYM result (1) precisely matches the Taylor coefficients of the corresponding D-instanton/SUGRA expression for G_n [4]. In our view, this highly nontrivial agreement, including the x_i dependence — taken together with the unexpected emergence of an $AdS_5\times S^5$ moduli space in large-N SYM — constitute convincing evidence in favor of Maldacena's conjecture.

We should emphasize that the above comparison between the SYM and SUGRA pictures can be quantitative if and only if there exists a nonrenormalization theorem that allows one to relate the small g^2N to the large g^2N behavior of chiral correlators such as G_n, as has been suggested in refs. [3, 5]. In the absence of such a theorem the best one can hope for is that qualitative features of the agreement persist beyond leading order while the exact numerical factor in each instanton sector does not, in analogy with the mismatch in the numerical prefactor between weak and strong coupling results for black-hole entropy [11]. In our view, however, our present results provide strong evidence in favor of such a nonrenormalization theorem for G_n, for the following reason. Consider the planar diagram corrections to the leading semiclassical (i.e. $g^2N\to 0$) result, eq. (1). In principle, these would not only modify the above result by an infinite series in g^2N, but also, at each order in this expansion, and independently for each value of k, they would produce a different function $F_n(x_1,\ldots,x_n)$. The fact that the leading semiclassical form for F_n that we obtain is not only k-independent, but already matches the D-instanton/SUGRA prediction exactly, suggests that such planar diagram corrections actually vanish.

This letter is a shortened version of [7], which will contain full calculational details, and will also show that the connection between D-instanton and SYM amplitudes is more general than the large-N context considered here. Maldacena's construction of the $\mathcal{N}=4$ theory starts from a configuration of N D3 branes on $\mathbb{R}^{10}$. The world-volume theory of the D-instanton has a Higgs branch which is exactly the ADHM moduli space of SU(N) [12]. In ref. [7], we will explain how the supersymmetric multi-instanton measure obtained in refs. [13, 14, 15], and used below, can be deduced from string theory directly.

In order to evaluate G_n in the SYM picture, one inserts the n appropriate gauge-invariant operators under the integration $\int d\mu^k_{\rm phys}\,\exp(-S^k_{\rm inst})$ where $S^k_{\rm inst}$ is the k-instanton action and $d\mu^k_{\rm phys}$ is the collective coordinate measure. These quantities are defined as follows [15]. The bosonic and fermionic collective coordinates live, respectively, in an $(N+2k)\times 2k$ complex matrix a, and in an $(N+2k)\times k$ Grassmann-valued complex matrix $\mathcal{M}^A$, where the SU$(4)_R$ index $A=1,2,3,4$ labels the supersymmetry.

In components:[1]

$$a = \begin{pmatrix} w_{ui\dot\alpha} \\ \left(a'_{\beta\dot\alpha}\right)_{li} \end{pmatrix}, \qquad \mathcal{M}^A = \begin{pmatrix} \mu^A_{ui} \\ \left(\mathcal{M}'^A_\beta\right)_{li} \end{pmatrix} \tag{2}$$

where both a'_m (defined by $a'_{\beta\dot\alpha} = a'_m \sigma^m_{\beta\dot\alpha}$) and $\mathcal{M}'^A_\beta$ are Hermitian $k \times k$ matrices. These matrices are subject to polynomial ADHM constraints discussed shortly, as well as to a $U(k)$ symmetry

$$w_{iu\dot\alpha} \to w_{ju\dot\alpha} U_{ji}\,, \qquad a'_{mij} \to U^{-1}_{ik}\, a'_{mkl}\, U_{lj}\,, \qquad U \in U(k)\,, \tag{3}$$

that must be modded out when constructing physical quantities. For future use we also introduce the boson bilinears

$$\left(W^{\dot\alpha}{}_{\dot\beta}\right)_{ij} = \bar w^{\dot\alpha}_{iu}\, w_{ju\dot\beta}\,, \qquad W^0 = \mathrm{tr}_2\,(W)\,, \qquad W^c = \mathrm{tr}_2\,(\tau^c W)\,, \qquad c = 1,2,3\,. \tag{4}$$

We remind the reader that in the dilute instanton gas limit, the diagonal elements of W^0 have the interpretation $W^0_{ii} = 2\rho_i^2$ where ρ_i is the scale-size of the i^{th} instanton; likewise $(-a'_m)_{ii}$ is its 4-position.

In terms of these quantities, and in the absence of adjoint VEVs, the $\mathcal{N} = 4$ k-instanton action is given by [17]

$$S^k_{\text{inst}} = \frac{8\pi^2 k}{g^2} + S^k_{\text{quad}}\,. \tag{5}$$

Here S^k_{quad} is a particular fermion quadrilinear term, with one fermion collective coordinate chosen from each of the four gaugino sectors $A = 1,2,3,4$:

$$S^k_{\text{quad}} = \frac{\pi^2}{g^2}\,\epsilon_{ABCD}\,\mathrm{tr}_k\left(\Lambda_{AB}\mathbf{L}^{-1}\Lambda_{CD}\right). \tag{6}$$

The $k \times k$ anti-hermitean fermion bilinear Λ_{AB} is defined by

$$\Lambda_{AB} = \frac{1}{2\sqrt{2}}\left(\bar{\mathcal{M}}^A\mathcal{M}^B - \bar{\mathcal{M}}^B\mathcal{M}^A\right), \tag{7}$$

whereas $\mathbf{L}$ is a linear self-adjoint operator that maps the k^2-dimensional space of such matrices onto itself:

$$\mathbf{L}\cdot\Omega = \frac{1}{2}\{\Omega, W^0\} - \frac{1}{2}\mathrm{tr}_2\left([\bar a', \Omega]a' - \bar a'[a', \Omega]\right). \tag{8}$$

[1] We adopt the conventions of refs. [15, 16] throughout; see [15] for references to the early literature on ADHM. The indices $u, v = 1, \ldots, N$ are SU(N) indices; $\alpha, \dot\alpha$, etc. $= 1,2$ are Weyl indices (traced over with 'tr_2'); $i, j = 1, \ldots, k$ (k being the topological number) are instanton indices (traced over with 'tr_k'); and $m, n = 1,2,3,4$ are Euclidean Lorentz indices. Integrals over $k \times k$ matrices are defined with respect to a hermitian basis T^r, $r = 1, \ldots, k^2$, normalized so that $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$.

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As discussed in ref. [17], $S^k_{\rm quad}$ is a SUSY invariant which lifts all the adjoint fermion modes except the 16 exact supersymmetric and superconformal modes.

As in refs. [15, 5], it will prove useful to replace the fermion quadrilinear $S^k_{\rm quad}$ with a fermion bilinear coupled to a set of auxiliary Gaussian variables. These take the form of an anti-symmetric tensor $\chi_{AB} = -\chi_{BA}$ whose elements are $k\times k$ matrices in instanton indices. The expression we need is

$$\exp\left(-S^k_{\rm quad}\right) = \pi^{-3k^2}\left(\det{}_{k^2}\mathbf{L}\right)^3 \int d^{6k^2}\chi\, \exp\Big[-\epsilon_{ABCD}{\rm tr}_k\left(\chi_{AB}\mathbf{L}\chi_{CD}\right) + \\ + 4\pi i g^{-1}{\rm tr}_k\left(\chi_{AB}\Lambda_{AB}\right)\Big]\,. \tag{9}$$

Requiring negative-definiteness of this Gaussian form is tantamount to the specific reality condition

$$\frac{1}{2}\epsilon_{ABCD}\chi_{CD} = \chi^\dagger_{AB}\,, \tag{10}$$

where the hermitian conjugation acts on instanton indices only. This reality condition simply means that χ_{AB} transforms in the vector representation of the SO(6) $\cong$ SU(4) R-symmetry. To make this manifest, we can introduce a 6-vector of $k\times k$ matrices χ_a, $a = 1,\ldots,6$ with

$$\chi_{AB} = \frac{1}{\sqrt{8}}\left(\eta^c_{AB}\chi_c + i\bar\eta^c_{AB}\chi_{c+3}\right), \tag{11}$$

where the η^c_{AB} ($\bar\eta^c_{AB}$), $c = 1,2,3$, are the (anti-)self-dual 't Hooft symbols [18]. Equation (10) then says that the six matrices χ_a are hermitian.

Next we turn to the k-instanton $\mathcal{N} = 4$ collective coordinate measure, which reads (see ref. [14] and sec. 6 of ref. [15]):

$$\int d\mu^k_{\rm phys} = \frac{2^{-k^2/2}(C''_1)^k}{{\rm Vol}\,U(k)}\int d^{2Nk}\bar w\, d^{2Nk}w\, d^{4k^2}a' \prod_{A=1,2,3,4} d^{Nk}\bar\mu^A\, d^{Nk}\mu^A\, d^{2k^2}\mathcal{M}'^A \\ \times\left(\det{}_{k^2}\mathbf{L}\right)^{-3}\prod_{r=1,\ldots,k^2}\Bigg[\prod_{c=1,2,3}\delta\left(\frac{1}{2}{\rm tr}_k\, T^r\left({\rm tr}_2\,\tau^c\bar a a\right)\right)\times \\ \times\prod_{A=1,2,3,4}\delta\left({\rm tr}_k\, T^r(\bar{\mathcal{M}}^A a + \bar a\mathcal{M}^A)\right)\Bigg] \tag{12}$$

The constant C''_1 is fixed at the 1-instanton level, by comparing eq. (12) with the standard 1-instanton 't Hooft-Bernard measure [18, 19]; in the Pauli-Villars scheme one obtains

$$C''_1 = 2^{-2N+1/2}\pi^{-6N}g^{4N}\,. \tag{13}$$

Note that the factors of $\det\mathbf{L}$ cancel out between eqs. (9) and (12). The matrix-valued arguments of the δ-functions in eq. (12) are the usual bosonic and fermionic ADHM constraints, respectively. An independent rederivation of this measure, directly from string theory, will be presented in ref. [7].

If, as in the present case, we intend to use our measure to integrate only gauge-invariant quantities (in particular, in the absence of adjoint VEVs), the expression (12) can be simplified by transforming to a smaller set of gauge-invariant collective coordinates (i.e. variables without an uncontracted 'u' index). In the bosonic sector this means changing variables from $\{w,\bar{w}\}$ to the W variables of eq. (4). An interesting jacobian identity is proved in ref. [7]:

$$\int_{\substack{\text{gauge}\\ \text{coset}}} d^{2kN}w\, d^{2kN}\bar{w} = c_{k,N}\left(\det{}_{2k}W\right)^{N-2k}\ d^{k^2}W^0 \prod_{c=1,2,3} d^{k^2}W^c\,. \tag{14}$$

where we have defined the constant

$$c_{k,N} = \frac{2^{2kN-4k^2+k}\pi^{2kN-2k^2+k}}{\prod_{i=1}^{2k}(N-i)!}\,. \tag{15}$$

We can already anticipate that as $N\to\infty$, the jacobian factor of $(\det W)^N = \exp(N\,\mathrm{tr}\log W)$ will be amenable to a saddle-point treatment. Another nice feature of this change of variables is that the bosonic ADHM constraints in eq. (12), which are quadratic in the $\{w,\bar{w}\}$ coordinates, become linear in terms of W; explicitly,

$$0 = W^c + [a'_n, a'_m]\,\mathrm{tr}_2\left(\tau^c\bar{\sigma}^{nm}\right) = W^c - i[a'_n, a'_m]\,\bar{\eta}^c_{nm}\,. \tag{16}$$

This means that we can eliminate the integrals over W^c from the measure.

Likewise in the fermion sector we change variables from $\{\mu,\bar{\mu}\}$ to $\{\zeta,\bar{\zeta},\nu,\bar{\nu}\}$ defined by

$$\mu^A_{iu} = w_{uj\dot{\alpha}}\,\zeta^{\dot{\alpha}A}_{ji} + \nu^A_{iu}\,, \qquad \bar{\mu}^A_{iu} = \bar{\zeta}^A_{\dot{\alpha}ij}\,\bar{w}^{\dot{\alpha}}_{ju} + \bar{\nu}^A_{iu}\,, \tag{17}$$

where the ν modes form a basis for the $\perp$-space of w:

$$0 = \bar{w}^{\dot{\alpha}}_{iu}\nu^A_{ju} = \bar{\nu}^A_{iu}w_{ju\dot{\alpha}}\,, \tag{18}$$

In these variables the fermionic ADHM constraints in eq. (12) have the gauge-invariant form

$$0 = \bar{\zeta}^A\,W + W\,\zeta^A + [\mathcal{M}'^A, a'] \tag{19}$$

which can be used to eliminate $\bar{\zeta}^A$ in favor of ζ^A and $\mathcal{M}'^A$; doing so gives a factor which precisely cancels the jacobian for the change of variables (17).

Notice that ν and $\bar\nu$ are absent from the constraint (19), and only appear in the coupling $\mathrm{tr}_k\,\chi_{AB}\Lambda_{AB}$ in eq. (9). We eliminate them from the measure, as follows. First, we decompose $\Lambda_{AB}=\hat\Lambda_{AB}+\tilde\Lambda_{AB}$, where

$$(\hat\Lambda_{AB})_{ij}=\frac{1}{2\sqrt2}\left(\bar\nu^A_{iu}\nu^B_{ju}-\bar\nu^B_{iu}\nu^A_{ju}\right)\,, \tag{20}$$

and

$$\tilde\Lambda_{AB}=\frac{1}{2\sqrt2}\Big(\bar\zeta^A W\zeta^B-\bar\zeta^B W\zeta^A+\{\mathcal{M}'^A\,,\,\mathcal{M}'^B\}\Big)\,. \tag{21}$$

Second, we calculate

$$\begin{aligned}\int\prod_{A=1,2,3,4}d^{k(N-2k)}\nu^A\,d^{k(N-2k)}\bar\nu^A\,\exp\left[\sqrt8\pi ig^{-1}\mathrm{tr}_k\,\chi_{AB}\bar\nu^A\nu^B\right]&=\\ =\left(\frac{8\pi^2}{g^2}\right)^{2k(N-2k)}&(\det{}_{4k}\chi)^{N-2k}\,,\end{aligned} \tag{22}$$

Note the similarity of this result to the jacobian in eq. (14); it too will contribute to the saddle-point equations in the large-N limit. The third and final contribution to these equations will be the Gaussian term in χ in eq. (9), once one rescales $\chi_{AB}\to\sqrt N\chi_{AB}$ so that N factors out in front. Combining the above manipulations, one now finds for the gauge-invariant measure:

$$\begin{aligned}\int d\mu^k_{\rm phys}\,e^{-S^k_{\rm inst}}=&\frac{g^{8k^2}2^{4kN-19k^2/2+k/2}e^{2\pi ik\tau}c_{k,N}}{\pi^{2kN+11k^2}\,\mathrm{Vol}\,\mathrm{U}(k)}\int d^{k^2}W^0\,d^{4k^2}a'\,d^{6k^2}\chi\,\times\\ &\times\prod_{A=1,2,3,4}d^{2k^2}\mathcal{M}'^A\,d^{2k^2}\zeta^A\,(\det{}_{2k}W\det{}_{4k}\chi)^{N-2k}\,\times\\ &\times\exp\Big[4\pi ig^{-1}\,\mathrm{tr}_k\,\chi_{AB}\tilde\Lambda^{AB}-\mathrm{tr}_k\,\chi_a\mathbf{L}\chi_a\Big]\,.\end{aligned} \tag{23}$$

where the "effective k-instanton action" is the sum of the three aforementioned terms (plus a constant piece):

$$S^k_{\rm eff}=-2k\,(1+3\log2)-\mathrm{tr}_{2k}\log W-\mathrm{tr}_{4k}\log\chi+\epsilon_{ABCD}\,\mathrm{tr}_k\,(\chi_{AB}\,\mathbf{L}\,\chi_{CD})\,. \tag{24}$$

We now formally pass to the limit $N\to\infty$, and stationarize $S^k_{\rm eff}$ with respect to the $11k^2$ variables a'_m, W^0 and χ_{AB}. The general solution to the coupled saddle-point equations reads (with no sum on i):

$$(W^{\dot\alpha}{}_{\dot\beta})_{ij}=\rho_i^2\delta_{ij}\delta^{\dot\alpha}{}_{\dot\beta}\,,\qquad(\chi_{AB})_{ij}=\frac{1}{\rho_i}\hat\Omega^i_{AB}\delta_{ij}\,,\qquad(a'_m)_{ij}=-X^i_m\delta_{ij}\,, \tag{25}$$

up to an adjoint action on all these quantities by the $\mathrm{U}(k)/\mathrm{U}(1)^k_{\rm diag}$ coset of the $\mathrm{U}(k)$ symmetry (3). For each value of the instanton index $i=1,\dots,k$, the quantity $\hat\Omega^i_{AB}$ satisfies

$$\epsilon_{ABCD}\hat\Omega^i_{AB}\hat\Omega^i_{CD}=1\qquad(\text{no sum on }i)\,. \tag{26}$$

Equivalently, in terms of the SO(6) vector variables (11), eqs. (25)-(26) imply that the six matrices χ_a are diagonal:

$$\chi_a = \mathrm{diag}\big(\rho_1^{-1}\hat{\Omega}_a^1, \dots, \rho_k^{-1}\hat{\Omega}_a^k\big)\,, \tag{27}$$

where $\hat{\Omega}_a^i$ is a unit length six-vector, related to $\hat{\Omega}_{AB}^i$ as per (11). In other words, up to $U(k)/\mathrm{U}(1)^k$, our k-instanton solution is parameterized by k points in the 10-dimensional space $AdS_5 \times S^5$, where, for each instanton, (X_m^i, ρ_i) are the coordinates in AdS_5 and $\hat{\Omega}_a^i$ is a point on S^5. One is ineluctably led to the conclusion that at large N the k-instanton measure in $\mathcal{N}=4$ SYM theory at the conformal point is related to a configuration of k D-instantons in string/SUGRA theory! Note also that the effective action (24) evaluated on the solution is zero.

This charge-k solution also has a very simple interpretation from the point of view of the gauge theory. As per our initial intuition, it consists of k individual SU(2) instantons, with positions X_m^i and scale sizes ρ_i, all embedded in mutually commuting SU(2) subgroups of the SU(N) gauge group. Indeed the embedding of the i^{th} instanton is determined by the vectors $w_{iu\dot\alpha}$, from which one forms the three SU(2) generators

$$(t^c)_{uv} = \frac{1}{\rho_i^2} w_{iu\dot\alpha}\,(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar{w}_{iv}^{\dot\beta} \qquad (\text{no sum on } i)\,. \tag{28}$$

The fact that, at the saddle point, $(W^{\dot\alpha}{}_{\dot\beta})_{ij} = 0$ for $i \neq j$, is precisely the statement that the two SU(2) embeddings commute.

The next step in the saddle-point expansion is to integrate the Gaussian fluctuations around the k-instanton solution. Here we encounter an important phenomenon which actually alters the naive N counting of the measure. When one analyses the quadratic fluctuations around the solution (25), one finds, generically, $10k$ zero modes corresponding to the parameters of the solution and $k(k-1)$ zero modes corresponding to the action of the $\mathrm{U}(k)/\mathrm{U}(1)^k$ symmetry (3). However, it can be shown that when pairs of instantons are at the same point in $AdS_5 \times S^5$ there are additional zeroes of the quadratic operator; the correspondingly singular small-fluctuations determinant then acts as an attractive singular potential on the instanton moduli space. As detailed in ref. [7], it turns out that such special solutions are more dominant in the large N limit and, indeed, the leading-order N dependence comes from the completely degenerate solution when *all* the instantons sit at the same point in $AdS_5 \times S^5$:

$$X_m^i \equiv X_m\,, \qquad \rho_i \equiv \rho\,, \qquad \hat{\Omega}_a^i \equiv \hat{\Omega}_a\,. \tag{29}$$

Notice that this special solution is completely invariant under U(k).

Around the special solution, the fluctuations fall into three sets. First, there are 10 true zero modes which correspond to the position of the k-instanton "bound state" in $AdS_5 \times S^5$. Second, there are k^2 fluctuations of the form

$$\varphi = 2\rho\epsilon_{ABCD}\hat{\Omega}_{AB}\delta\chi_{CD} + \frac{1}{2\rho^2}\delta W^0\,, \tag{30}$$

which have a nonzero quadratic coefficient in the small-fluctuations expansion. The remaining $10k^2 - 10$ fluctuations first appear beyond quadratic order and they correspond to the traceless parts $\hat{\chi}_a$ and $\hat{a}'_m$ of the ten $k \times k$ matrices χ_{AB} and a'_m.

It turns out that in the expansion of the effective action around the special solution, there is no cubic coupling involving three variations of $\hat{a}'_m$ and $\hat{\chi}_a$. This means that at leading order in N

$$\varphi \sim \mathcal{O}\left(N^{-1/2}\right), \qquad \hat{a}'_m \sim \mathcal{O}\left(N^{-1/4}\right), \qquad \hat{\chi}_{AB} \sim \mathcal{O}\left(N^{-1/4}\right). \tag{31}$$

The Gaussian fluctuations φ can be integrated out explicitly, leaving an additional contribution to the quartic coupling of the fluctuations $\hat{a}'_m$ and $\hat{\chi}_{AB}$. To complete the expansion, the fermion terms in (24) involve the traceless parts $\hat{\zeta}^{\dot{\alpha}A}$ and $\hat{\mathcal{M}}'^{A}_{\alpha}$ coupled to $\hat{a}'_m$ and $\hat{\chi}_{AB}$.

Remarkably, in the large-N limit, the leading-order terms of the effective action around the special solution, with the quadratic fluctuations φ integrated out, precisely assemble themselves into the dimensional reduction from ten to zero of $\mathcal{N} = 1$ supersymmetric Yang-Mills with gauge group SU(k) in *flat* space. The SU(k) adjoint-valued ten-dimensional gauge field and Majorana-Weyl fermion are defined in terms of the fluctuations:

$$A_\mu = N^{1/4}\left(\rho^{-1}\hat{a}'_m, \rho\hat{\chi}_a\right), \qquad \Psi = \left(\frac{\pi}{2g}\right)^{1/2} N^{1/8}\left(\rho^{-1/2}\hat{\mathcal{M}}'^{A}_{\alpha}, \rho^{1/2}\hat{\zeta}^{\dot{\alpha}A}\right). \tag{32}$$

The action for the dimensionally reduced gauge theory is

$$S(A_\mu, \Psi) = -\frac{1}{2}\mathrm{tr}_k\,\left[A_\mu, A_\nu\right]^2 \;+\; \mathrm{tr}_k\,\bar{\Psi}\Gamma_\mu\left[A_\mu, \Psi\right]. \tag{33}$$

It might have been anticipated that the action would depend on $\hat{\Omega}_a$ in some way. Actually it does, but only through the representation of the ten-dimensional gamma matrices Γ_μ [7].

We conclude that the effective gauge-invariant measure for k instantons in the large-N limit factorizes into a 1-instanton-like piece, for the position of the bound state in $AdS_5 \times S^5$ and the 16 supersymmetric and superconformal modes, times the partition function $\mathcal{Z}_k$ of the dimensionally-reduced $\mathcal{N} = 1$ supersymmetric SU(k) gauge theory in flat space:

$$\int d\mu^k_{\mathrm{phys}}\, e^{-S^k_{\mathrm{inst}}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8 e^{2\pi i k\tau}}{k^3 2^{17k^2/2-k/2+25}\pi^{9k^2/2+9}} \int \rho^{-5}d\rho\, d^4X\, d^5\hat{\Omega} \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A \cdot \hat{\mathcal{Z}}_k\,, \tag{34}$$

with

$$\hat{\mathcal{Z}}_k = \frac{1}{\mathrm{Vol}\,\mathrm{SU}(k)} \int_{\mathrm{SU}(k)} d^{10}A\, d^{16}\Psi\, e^{-S(A_\mu,\Psi)}\,, \tag{35}$$

$$S(A_\mu,\Psi) = N(S_\mathrm{b}+S_\mathrm{f}) = -\frac{1}{2}\mathrm{tr}_k\,[A_\mu,A_\nu]^2 + \mathrm{tr}_k\,\bar\Psi\Gamma_\mu\,[A_\mu,\Psi]\,. \tag{36}$$

and where the appropriate group volume factor is

$$\mathrm{Vol}\,(\mathrm{SU}(k)) = \frac{2^{k-1}\pi^{k(k+1)/2-1}}{\prod_{i=1}^{k-1} i!}\,. \tag{37}$$

When integrating $\mathrm{SU}(k)$ singlet quantities, which is relevant for the comparison to string theory, the factor $\hat{\mathcal{Z}}_k$ is simply a constant. It was known [8, 9] that this constant is proportional to $\sum_{d|k} d^{-2}$ (a sum over the integer divisors of k); however, the constant of proportionality has only been recently settled [10]:

$$\hat{\mathcal{Z}}_k = 2^{17k^2/2-k/2-8}\pi^{9k^2/2-9/2}k^{-1/2}\sum_{d|k}\frac{1}{d^2}\,. \tag{38}$$

Using this expression, our final expression for the measure on gauge invariant and $\mathrm{SU}(k)$ singlet operators in the large N limit is

$$\int d\mu^k_\mathrm{phys}\, e^{-S^k_\mathrm{inst}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^8}{2^{33}\pi^{27/2}}\, k^{-7/2}e^{2\pi i k\tau}\sum_{d|k}\frac{1}{d^2}\int \frac{d^4X\,d\rho}{\rho^5}\,d^5\hat\Omega \prod_{A=1,2,3,4} d^2\xi^A d^2\bar\eta^A\,. \tag{39}$$

Finally, we can use our measure to calculate the correlation functions $G_n(x_1,\ldots,x_n)$. This entails inserting into eq. (39) the appropriate product of gauge-invariant composite chiral operators $\Phi_1(x_1)\times\cdots\times\Phi_n(x_n)$, which together contain the requisite 16 exact fermion modes, and approximating each Φ_j by its k-instanton saddle-point value $\Phi_j^{(k)}$. Explicit forms for the Φ_j are given in ref. [4], and need not be repeated here. However, we can make the following observation. Since, at leading order in N, the k instantons sit at the same point in $AdS_5\times S^5$, albeit in mutually commuting $\mathrm{SU}(2)$ subgroups of $\mathrm{SU}(N)$, it follows that $\Phi_j^{(k)}$ is simply proportional to its single-instanton counterpart: $\Phi_j^{(k)} = k\Phi_j^{(1)}$. Therefore G_n scales like k^n. Furthermore, the spatial function $F_n(x_1,\ldots,x_n)$ is precisely the same as in the 1-instanton sector, where its reinterpretation as a convolution of SUGRA propagators, as dictated by Maldacena's conjecture, has already been emphasized in ref. [4]. The final result of our Yang-Mills instanton calculation is summarized in eq. (1), where we absorb the constant factor into F_n.

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